THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.; E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,

AND BOMBAY.

VOL. II. JULY, 1903. No. 40.

A CHAPTER ON ALGEBRA.

I., II., III. On Inequalities.
IV. The Logarithm.
V. On Averages.
VI. The Logarithmic and other Expansions.
VII. The Exponential Expansion for Commensurable Indices.
VIII. General Binomial Expansion.
IX. On Incommensurable Indices.
X. On the same Theorems for a Vector Variable.

THE aim of the following articles is to meet certain difficulties and defects in the accepted book-work of the "Higher Algebra" of school work, especially in connection with infinite series.

In the book-work of Inequalities the generalization from the binomial case seems unsatisfactory.

In the treatment of certain fundamental infinite series, I have tried, by obtaining simple limits of error, to make the 'expansion to infinity' as intelligible as it is in the case of Geometrical Progression, without having recourse to theorems on convergence. The exponential expansion I have established without the use of the general Binomial, and have avoided the use of the incommensurable index, which is usually introduced undefined.

In proving the Binomial Theorem for any index I have avoided the multiplication of infinite series.

"The logarithm" I have ventured to define as a function independent of e. This definition involves no interference with the usual definition of a "logarithm to a given base," in which the word is used merely as a synonym for "index."

I.

* *The Arithmetic Mean of any number of positive quantities is greater than the Geometric Mean.*

(a) Let a', b' be any two quantities having the same sum as a, b; so that

$$a'+b'=a+b.$$

Then $$a'b'-ab=a'(a+b-a')-ab=-(a'-a)(a'-b),$$

which is positive if a' lies between a and b.

[Since $a'+b'=a+b$, b' will in this case also lie between a and b.]

* [While these articles are in print I find from the May number of the Gazette that this proof has been published previously by Mr. G. E. Crawford.]

Therefore a product of two quantities is increased by changing to two quantities having the same sum but lying between the former two.

(β) Let $a, b, c \ldots$ be n quantities: and let μ be their arithmetic mean, so that $$n\mu = a+b+c+\ldots .$$

Now μ certainly lies between some two of the quantities $a \,.\, b \,.\, c \ldots$.
If $a \,.\, b$ are two such quantities.

$$ab < \mu(a+b-\mu);$$

$$\therefore\ abc \ldots < \mu(a+b-\mu)cd \ldots$$

Now $$(a+b-\mu)+c+d+\ldots = (n-1)\mu;$$

$$\therefore\ \mu \text{ is the A.M. of the } \overline{n-1} \text{ quantities } a+b-\mu, c, d \ldots,$$

and therefore the product $(a+b-\mu)cd \ldots < \mu \times \overline{n-2}$ quantities, whose mean is again μ.
By repetition of this process we obtain the result

$$abc \ldots < \mu\mu\mu \ldots .$$

or $$\mu > (abc \ldots)^{\frac{1}{n}}* \qquad \text{Q.E.D.}$$

NOTE. †The theorem $\dfrac{a^i+b^i+\ldots}{n} > \left(\dfrac{a+b+\ldots}{n}\right)^i$ can be proved by a similar method, when i is an integer.

II.

If "Av. a^m" stands for the average of the m^{th} powers of a number of positive quantities, i.e. for $\dfrac{a^m+b^m+\ldots}{n}$, *then*

$$|\,Av.\ a^m\,|^{\frac{1}{m}} > |\,Av.\ a^{m'}\,|^{\frac{1}{m'}} \text{ provided } (m-m') \text{ is positive.}$$

Consider $\dfrac{a^m+b^m+\ldots}{a^{m-1}+b^{m-1}+\ldots} - \dfrac{a^{m-1}+b^{m-1}+\ldots}{a^{m-2}+b^{m-2}+\ldots}$ where m has any value.

This difference varies as

$$\Sigma\,|\,a^m b^{m-2} + b^m a^{m-2} - 2a^{m-1}b^{m-1}\,|\,,$$

the summation extending to every pair of letters.
This sum becomes

$$\Sigma a^{m-2}b^{m-2}(a-b)^2, \text{ which is positive.}$$

$$\therefore\ \frac{\Sigma a^m}{\Sigma a^{m-1}} > \frac{\Sigma a^{m-1}}{\Sigma a^{m-2}} \text{ for all values of } m;$$

$$\therefore\ \frac{\Sigma a^i}{\Sigma a^{i-1}} > \frac{\Sigma a^{i-1}}{\Sigma a^{i-2}} > \ldots > \frac{\Sigma a^j}{\Sigma a^{j-1}} \ldots > \frac{\Sigma a^1}{\Sigma a^0} > \frac{\Sigma a^0}{\Sigma a^{-1}} \ldots > \frac{\Sigma a^{-k+1}}{\Sigma a^{-k}} \ \ldots$$

where i, j, k are positive integers and $i > j$.

---

*Throughout these articles fractional indices are attached to positive quantities only and imply the simple positive root only, so that no ambiguity arises.

† I have followed Thomson and Tait's use of i and j for integers.

Hence it follows, by taking products of successive fractions, that

$$\left(\frac{\Sigma a^i}{\Sigma a^0}\right)^j > \left(\frac{\Sigma a^j}{\Sigma a^0}\right)^i > \left(\frac{\Sigma a^1}{\Sigma a^0}\right)^{ij} > \left(\frac{\Sigma a^0}{\Sigma a^{-k}}\right)^{\frac{i}{k}};$$

$$\therefore \left(\frac{\Sigma a^i}{n}\right)^{\frac{1}{i}} > \left(\frac{\Sigma a^j}{n}\right)^{\frac{1}{j}} > \frac{\Sigma a}{n} > \left(\frac{\Sigma a^{-k}}{n}\right)^{-\frac{1}{k}};$$

Therefore, putting $a^j \equiv a^{m'}$, where m' has any value, and writing m for $\frac{i}{j}$ and m'' for $\frac{k}{j}$, we obtain

$$(\text{Av. } a^m)^{\frac{1}{m}} > (\text{Av. } a^{m'})^{\frac{1}{m'}} > (\text{Av. } a^{-m''})^{-\frac{1}{m''}},$$

i.e. $(\text{Av. } a^m)^{\frac{1}{m}}$ moves up and down the scale with m. Q.E.D.

NOTE. By a similar method of proof the following theorem can be established. *If "Av. S_r" stands for the average value of the products r together of n quantities, then $\{Av.\ S_r\}^{\frac{1}{r}}$ diminishes as r increases.*

III.

Applications.

If in § I. we take p of the n quantities to be x and the rest 1, we obtain

$$\frac{px+(n-p)}{n} > (x^p)^{\frac{1}{n}};$$

$\therefore$ putting $\frac{p}{n} \equiv f$, so that f is a positive proper fraction,

$$x^f < 1+f(x-1).$$

Now write $x^{\frac{1}{f}}$ for x,

Then $$x < 1+f(x^{\frac{1}{f}}-1);$$

$$\therefore x^{\frac{1}{f}} > 1+\frac{1}{f}(x-1);$$

$\therefore$ if n is positive $x^n \gtrless 1+n(x-1)$ according as $n \gtrless 1$. .....(*a*)

Again, writing $\frac{1}{x}$ for x: $x^{-n} \gtrless 1+n\left(\frac{1}{x}-1\right)$ as $n \gtrless 1$;

$\therefore x^{1-n} \gtrless x-n(x-1)$, *i.e.* $1+(1-n)(x-1)$, as $n \gtrless 1$;

$\therefore$ for all positive values of n, $x^{-n} > 1-n(x-1)$;

$\therefore x^n \gtrless 1+n(x-1)$, according as n is outside or within the limits 0, 1.

Again, putting $\frac{1}{x}$ for x:

$$\left.\begin{aligned} &\frac{1}{x^n} \lessgtr 1+n\left(\frac{1}{x}-1\right) \\ &\therefore\ 1 \gtrless x^n - nx^{n-1}(x-1) \\ &\therefore\ x^n \lessgtr 1+n(x-1).x^{n-1} \end{aligned}\right\} \text{ for the same limits of } n.$$

Hence, *for all values of n,*

x^n lies between $1+n(x-1)x^{n-1}$ and $1+n(x-1)$.

IV.

The Logarithm.

If n is any positive commensurable number, we may write it $\frac{I}{J}$ where I and J may be supposed indefinitely great provided their ratio is unaltered;

$$\therefore\ x^n=\left(x^{\frac{1}{J}}\right)^I.$$

Now let $$x^{\frac{1}{J}}\equiv 1+\frac{X}{J}.$$

We proceed to investigate the value of X:

(1) By § III. $$x^{\frac{1}{J}}<^{*}\left|\begin{matrix}1+\dfrac{x-1}{J}\\ 1+\dfrac{x-1}{J}(x)^{\frac{1}{J}-1}\end{matrix}\right.$$

$$\therefore\ X<\left|\begin{matrix}x-1\\ (x-1)x^{\frac{1}{J}-1}\end{matrix}\right. \text{ and } \therefore\ <\left|\begin{matrix}x-1\\ \dfrac{x-1}{x}.\end{matrix}\right.$$

(2) $$X=J(x^{\frac{1}{J}}-1).$$

Now if $$J'>J,$$

$$J'(x^{\frac{1}{J'}}-1)\gtrless J(x^{\frac{1}{J}}-1)$$

according as $$x^{\frac{1}{J'}}\gtrless 1+\frac{J}{J'}(x^{\frac{1}{J}}-1).$$

Now by § III., $$(x^{\frac{1}{J}})^{\frac{J}{J'}}<1+\frac{J}{J'}(x^{\frac{1}{J}}-1);$$

$$\therefore\ X \text{ diminishes as } J \text{ increases;}$$

$$\therefore \text{ the value of } X \text{ given by } x^{\frac{1}{J}}=1+\frac{X}{J}$$

lies between certain finite limits, and as J increases indefinitely, diminishes continuously down to a certain fixed value.

This definite value is called the logarithm of x.

(3) Let $$x^{\frac{1}{J}}\equiv 1+\frac{X}{J},\quad y^{\frac{1}{J}}\equiv 1+\frac{Y}{J},\quad z^{\frac{1}{J}}\equiv 1+\frac{Z}{J}.$$

Then if $z=xy$, $$1+\frac{Z}{J}=\left(1+\frac{X}{J}\right)\left(1+\frac{Y}{J}\right);$$

$$\therefore\ Z=X+Y+\frac{XY}{J}.$$

Now when J is indefinitely great, X, Y, Z become $\log x$, $\log y$, $\log z$;

$$\therefore\ \log(xy)=\log x+\log y.$$

Hence $$\log\frac{x}{y}=\log x-\log y,\quad \log 1=0,\quad \log\frac{1}{x}=-\log x,$$

and $$\log x^{\pm\frac{i}{j}}=\pm\frac{i}{j}\log x=\pm\frac{i}{j}\log x^j;$$

therefore, writing x for x^j and n for $\left(\pm\frac{i}{j}\right)$,

$\log x^n=n\log x$ for all commensurable values of n.

---

* This notation is used to represent "lies between."

V.

On Averages.

Since $x^n < \dfrac{1+n(x-1)x^{n-1}}{1+n(x-1)}$, we have, writing $\left(1+\dfrac{b}{a}\right)$ for x,

$$(a+b)^n - a^n < \frac{nb(a+b)^{n-1}}{nba^{n-1}}$$

Now consider the increment b divided into N equal parts, so that each part is $\dfrac{b}{N}$.

For brevity write a_i for $a+\dfrac{i}{N}b$.

Then, by the above inequality, $a_i^n - a_{i-1}^n < \dfrac{n\frac{b}{N}a_i^{n-1}}{n\frac{b}{N}a_{i-1}^{n-1}}$

and taking the sum of all such differences over the whole of b,

$$(a+b)^n - a^n < \frac{\frac{nb}{N}\sum_1^N a_i^{n-1}}{\frac{nb}{N}\sum_0^{N-1} a_i^{n-1}}$$

Now the difference between these limiting values is $\dfrac{nb}{N}\{(a+b)^{n-1}-a^{n-1}\}$, and since N is independent of all the other quantities under consideration, we may suppose N indefinitely great and neglect this difference.

This limiting value of $\dfrac{1}{N}\sum_0^N a_i^{n-1}$ we call 'the average' of the continuous scale represented by x^{n-1} as x varies from a to $a+b$, and write

$$(a+b)^n - a^n = nb \,.\, \underset{\theta=0}{\overset{\theta=1}{\text{Av.}}}(a+\theta b)^{n-1}$$

$\therefore$ putting $a=0$ and $b=1$, $\underset{0}{\overset{1}{\text{Av.}}}\theta^{n-1} = \dfrac{1}{n}$ (at least if $n>1$);

$\therefore$ $\underset{0}{\overset{1}{\text{Av.}}}\theta = \frac{1}{2}$. $\underset{0}{\overset{1}{\text{Av.}}}\theta^2 = \frac{1}{3}$, etc.

VI.

The Expansion of the Logarithm.

Since $\log x < \left| \dfrac{x-1}{1-x^{-1}} \right.$

$$\log(a+b) - \log a < b \left| \frac{\frac{1}{a}}{\frac{1}{a+b}} \right.$$

Therefore, dividing b as before into N parts, and writing a_i for $\left(a+\dfrac{ib}{N}\right)$,

$$\log a_i - \log a_{i-1} < \frac{b}{N} \left| \frac{\frac{1}{a_{i-1}}}{\frac{1}{a_i}} \right.$$

$$\therefore \text{ adding, } \log(a+b)-\log a < \frac{b}{N}\left|\begin{matrix}\sum_0^{N-1}\frac{1}{a_i}\\ \sum_0^{N}\frac{1}{a_i}\end{matrix}\right.$$

$$\therefore = b \underset{0}{\overset{b}{\text{Av.}}} \frac{1}{a+x} \text{ when } N \text{ is very great.}$$

Now $$\frac{1}{a-x}=a^{-1}+a^{-2}x+\ldots+a^{-i-1}x^i+\frac{a^{-i-1}x^{i+1}}{a-x};$$

$$\therefore \log(a+b)-\log a = b \underset{0}{\overset{b}{\text{Av.}}} \left|\sum_0^i a^{-i-1}x^i(-)^i+\frac{(-)^{i+1}a^{-i-1}x^{i+1}}{a+x}\right|$$

but, by § V., $$\underset{0}{\overset{b}{\text{Av.}}}\, x^i \equiv \frac{b^i}{i+1};$$

$$\therefore \log(a+b)-\log a = \frac{b}{a}-\frac{b^2}{2a^2}\ldots(-)^i\frac{b^{i+1}}{a^{i+1}}+(-)^{i+1}\frac{b}{a^{i+1}}\underset{0}{\overset{b}{\text{Av.}}}\frac{x^{i+1}}{a+x},$$

and the error after i terms is

$$(-)^i\frac{b}{a^i}\underset{0}{\overset{b}{\text{Av.}}}\frac{x^i}{a+x} \text{ or } (-)^i\frac{b^{i+1}}{a^{i+1}}\underset{0}{\overset{1}{\text{Av.}}}\frac{\theta^i}{1+\frac{b}{a}\theta},$$

which vanishes when $\frac{b}{a}<1$ numerically, and when $\frac{b}{a}=+1$.

The same method of proof can be used to establish the Exponential and Binomial Expansions.

Thus, for the Exponential, since $X < \dfrac{x-1}{1-\frac{1}{x}}$

$$\therefore x < \frac{1+Xx}{1+X} \text{ and } x^n < \frac{1+nXx^n}{1+nX}$$

$$\therefore \text{ as above, } x = 1+X\underset{\theta=0}{\overset{\theta=1}{\text{Av.}}}x^\theta$$

and finally, $x=1+X+\frac{X^2}{\lfloor 2}+\ldots+\frac{X^i}{\lfloor i}+$ a remainder which can be easily shown to lie between $\frac{X^{i+1}}{\lfloor i+1}\cdot\left|\frac{x}{1}\right|$.

For the Binomial, since $x^n < \dfrac{1+n(x-1)x^{n-1}}{1+n(x-1)}$

$$\therefore x^n = 1+n(x-1).\underset{\theta=0}{\overset{\theta=1}{\text{Av.}}}\{1+\theta(x-1)\}^{n-1};$$

whence, writing $(n)_i$ for $\dfrac{n(n-1)\ldots(n-i+1)}{\lfloor i}$,

$x^n=1+n(x-1)+(n)_2.(x-1)^2+\ldots+(n)_i.(x-1)^i+$ a remainder which can be shown to lie between $(n-1)_i(x-1)^i\left|\begin{matrix}x^n-1\\ 0\end{matrix}\right|$.

The drawback to these proofs is that they are not readily extended to the case of vector values of x. Therefore in the succeeding articles other proofs are given which hold for all values of x.

(To be continued.)

NOTE ON THE BITANGENTS OF A PLANE CURVE.

The equation of the curve of order 14, on which lie the points of contact of the 28 bitangents of a quartic curve, was first given by Hesse (*Crelle's Journal*, Vols. 36, 40, 41). A very concise form of the equation in the Clebsch-Aronhold symbols was obtained by Dersch (*Math. Ann.*, Bd. 7). The method adopted in his Memoir was practically that given by Salmon (*Phil. Mag.*, Oct. 1858), and followed by Cayley (*Phil. Trans.*, 1859).

In 1886 appeared Maisano's paper (*Math. Ann.*, Bd. 29) containing the equation to the curve on which lie the contacts of the bitangents of the general quintic curve.

In the present Note, attention is drawn to a certain covariant curve belonging to the plane curve considered.

Let $f(x, y, z)=0$ be the equation in homogeneous point-coordinates to a curve of order n, which we shall assume non-singular.

A point (x', y', z') has, with respect to f, a set of polar curves

$$\Omega f=0,$$
$$\Omega^2 f=0,$$
$$\cdots\cdots$$
$$\Omega^r f=0,$$
$$\cdots\cdots$$

where $$\Omega \equiv x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z},$$

called the first, second, ... rth, ... polar curves of $x'y'z'$.

The $(n-1)$th polar curve of a point is a straight line ; and every line has $(n-1)^2$ poles.

The covariant curve under discussion is the locus of the poles of the tangent lines to the given curve f.

One of the $(n-1)^2$ poles of the tangent to f at the point P is evidently the point P itself, so that the ground curve f is part of the locus. The remainder of the locus constitutes a covariant curve $\phi=0$ of order $n^2(n-2)$, cutting f in its bi-contact points, and having inflexions with the same tangent line at the inflexions of f—thus verifying the number of bi-contacts of f from the identity

$$2\,.\,\tfrac{1}{2}n(n-2)(n^2-9)+3\,.\,3n(n-2)=n\,.\,n^2(n-2).$$

We proceed to prove these theorems.

The equation $\phi=0$ can be found by writing $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ for the line-coordinates in the tangential equation of f, and dividing out the factor f.

The tangential equation of the non-singular curve f is of order $n(n-1)$ in the line-coordinates. Substituting $\frac{\partial f}{\partial x}$, ... which are of order $(n-1)$ in the point-coordinates, and dividing out f, the remaining locus is of order

$$n(n-1)^2-n=n^2(n-2).$$

A bitangent to f, having contact at P and Q, has P, Q for two of its $(n-1)^2$ poles. Hence P, corresponding to the tangent at Q, lies on ϕ. Similarly for all the bi-contacts.

Let O be a point of inflexion on f. Referred to the tangent and normal at O as x, y axes, the equation of f in the neighbourhood of O takes the form

$$y=\alpha x^3+\beta x^4+\text{higher powers}.$$

The polar line of x', y' is

$$x\{3\alpha x''^2+4\beta x''^3+\ldots\}-y+\{-(n-1)y'+\alpha(n-3)x''^3+\ldots\}=0.$$

The tangent-line at x'', y'', a point on f very near O, is

$$x\{3\alpha x''^2+4\beta x''^3+\ldots\}-y+\{-2\alpha x''^3+\ldots\}.$$

Now since an inflexion may be regarded as the limiting point of coincidence of the two contacts of a bitangent, the curve ϕ must pass through O; and the polar line of a point x', y' on ϕ very near O will coincide with the tangent at x'', y'' on f near O, to the first order of smallness, if

$$x'^2=x''^2 \text{ and } (n-1)y'-(n-3)\alpha x''^3=2\alpha x''^3.$$

Rejecting the solution $x'=x''$ corresponding to the point of contact, the pole of the line near its contact is given by

$$x'=-x'', \ y'=-\frac{n-5}{n-1}\alpha x''^3,$$

to the first order of small quantities. Hence the curve ϕ in the neighbourhood of O approximates to the form of the curve

$$y=\frac{n-5}{n-1}\alpha x^3, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

in other words has an inflexion at O with the same tangent there as f.

It may be verified without difficulty that the equation of ϕ for the cubic

$$f\equiv x^3+y^3+z^3+6mxyz$$

is of the form

$$\Sigma x^9+6mxyz\,\Sigma x^6+\alpha\,.\,\Sigma(y^3z^6+y^6z^3)+\beta\,.\,x^2y^2z^2\,.\,\Sigma x^3+\gamma\,.\,x^3y^3z^3,$$

where α, β, γ are numerical, a curve of order 9 having inflexions at all those of the cubic.

It may be remarked that if $C=0$ be the equation to the curve found by Hesse, Salmon, and others, the above curve ϕ can be written

$$\phi\equiv Af+CH^3,$$

where A is a quantic of order $n^2(n-2)-n$.

We can connect ϕ with the Steinerian of f as follows:

The Steinerian is a curve of order $3(n-2)^2$, class $3(n-1)(n-2)$, which is the locus of a point whose first polar for f has a node—and is also the envelope of line-polars of points of the Hessian. .....(1)

It is known that the Steinerian touches all the inflexional tangents to f....(2), and by (1), if the point (x, y, z) lies on the Hessian, the line whose coordinates are

$$l : m : n=\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}$$

touches the Steinerian.

Suppose now (x, y, z) lies also on ϕ. Then the line

$$l : m : n=\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}$$

touches f. Hence to every intersection of ϕ and H corresponds a common tangent to f and the Steinerian S.

Now the inflexional tangents to f are common tangents to f and S by (2), and since such a tangent to f is a stationary tangent, each must count twice as a common tangent to f and S. Thus since ϕ, H have $3n^2(n-2)^2$ common points, f and S have $3n^2(n-2)^2+3n(n-2)$ common tangents. This number $=3n(n-2)(n-1)^2$, and f is of class $n(n-1)$. Hence the Steinerian must be of class $3(n-1)(n-2)$, verifying a known result. A. P. THOMPSON.

EXTENSION OF HUYGENS' APPROXIMATION TO A CIRCULAR ARC.

HUYGENS' approximation * to the length of a circular arc is $\frac{1}{3}(8l_1 - l_0)$, where l_0 is the chord of the arc and l_1 is the chord of half the arc. This result is most easily obtained from the series for $\sin\left(\frac{L}{2R}\right)$ and $\sin\left(\frac{L}{4R}\right)$ by neglecting powers of $\frac{L}{R}$ beyond the third, L being the true length of the arc and R the radius of the circle. For arcs less than 45° Huygens' rule is very accurate, the error for an arc of 30° being less than 1 in 300,000.

The general form of the approximation when powers of $\frac{L}{R}$ beyond the $(2n-1)^{\text{th}}$ are neglected is given by the equation:

$$L_n(a_1a_2a_3 \dots a_n) = 2^{b_n}l_n - \frac{a_n}{a_1}2^{b_{n-1}}l_{n-1} + \frac{a_na_{n-1}}{a_1a_2}2^{b_{n-2}}l_{n-2}$$

$$\dots(-1)^{n-1}\frac{a_n}{a_1}2^{b_1}l_1 + (-1)^n l_0,$$

where $a_1a_2a_3 \dots a_n$ denote $4-1,\ 4^2-1,\ 4^3-1, \dots 4^n-1$;

$b_1b_2b_3 \dots b_n$ denote $2^2-1,\ 3^2-1,\ 4^2-1, \dots (n+1)^2-1$;

$l_0l_1l_2 \dots l_n$ denote the lengths of the *chords* of the arc,

$$\frac{1}{2}\text{ arc},\ \frac{1}{4}\text{ arc}, \dots \frac{1}{2^n}\text{ arc},$$

and L_n is the required approximation.

The following are the first six approximations:

$$L_0 = l_0,$$

$$3L_1 = 2^3l_1 - l_0,$$

$$3.15L_2 = 2^8l_2 - 5.2^3l_1 + l_0,$$

$$3.15.63L_3 = 2^{15}l_3 - 21.2^8l_2 + 21.2^3l_1 - l_0,$$

$$3.15.63.255L_4 = 2^{24}l_4 - 85.2^{15}l_3 + 357.2^8l_2 - 85.2^3l_1 + l_0,$$

$$3.15.63.255.1023L_5 = 2^{35}l_5 - 341.2^{24}l_4 + 5797.2^{15}l_3 - 5797.2^8l_2 + 341.2^3l_1 - l_0.$$

If the proof is given for the approximation L_3 there will be no difficulty in detecting its generality on account of the symmetry of the determinants employed.

From the expansion of $\sin\theta$ in terms of θ we have:

$$\frac{l_0}{2R} = \sin\frac{L}{2R} = \frac{L}{2R} - \frac{1}{3!}\frac{L^3}{2^3R^3} + \frac{1}{5!}\frac{L^5}{2^5R^5} - \frac{1}{7!}\frac{L^7}{2^7R^7}$$

or

$$l_0 = L\left[1 - \frac{1}{3!}\frac{L^2}{2^2R^2} + \frac{1}{5!}\frac{L^4}{2^4R^4} - \frac{1}{7!}\frac{L^6}{2^6R^6} \dots\right].$$

---

*Huygens published his approximation in 1654 in his "*De Circuli Magnitudine Inventa*" in the following form: "Any arc less than a semicircle is *greater* than its subtense together with one third of the difference, by which the subtense exceeds the sine, and *less* than the subtense together with a quantity, which is to the said third, as four times the subtense added to the sine is to twice the subtense with three times the sine." If we apply the former part of this proposition to half the arc we obtain the rule as generally stated.

Writing down the similar expansions for l_1, l_2 and l_3, neglecting powers of $\frac{L}{R}$ beyond the sixth in each case, we can immediately eliminate

$$\frac{L^2}{3!\,R^2},\quad \frac{L^4}{5!\,R^4} \text{ and } \frac{L^6}{7!\,R^6}$$

in the form of a determinant,

$$\begin{vmatrix} l_0 - L & 2^{-2} & 2^{-4} & 2^{-6} \\ l_1 - \frac{L}{2} & 2^{-5} & 2^{-9} & 2^{-13} \\ l_2 - \frac{L}{4} & 2^{-8} & 2^{-14} & 2^{-20} \\ l_3 - \frac{L}{8} & 2^{-11} & 2^{-19} & 2^{-27} \end{vmatrix} = 0,$$

and calling the value of L obtained from this equation L_3 we have

$$L_3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2^{-1} & 2^{-3} & 2^{-5} & 2^{-7} \\ 2^{-2} & 2^{-6} & 2^{-10} & 2^{-14} \\ 2^{-3} & 2^{-9} & 2^{-15} & 2^{-21} \end{vmatrix} = \begin{vmatrix} l_0 & 1 & 1 & 1 \\ l_1 & 2^{-3} & 2^{-5} & 2^{-7} \\ l_2 & 2^{-6} & 2^{-10} & 2^{-14} \\ l_3 & 2^{-9} & 2^{-15} & 2^{-21} \end{vmatrix}$$

These determinants may be reduced immediately if it is remembered that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

In expanding the determinant of the left-hand member of the equation in this way, the power of 2 which may be taken out as a factor will be $2^{-3.1-5.2-7.3} = 2^{-34}$. So the powers of 2 which may be extracted as factors from the minors of l_0, l_1, l_2, and l_3 of the right-hand member are 2^{-34}, 2^{-34+3}, $2^{-34+3+5}$, $2^{-34+3+5+7}$ respectively.

The equation may then be written:

$$\begin{aligned} L_3 . 2^{-34}(2^2-1)(2^4-1)(2^6-1)(2^2-1)(2^4-1)(2^2-1) &= -l_0 2^{-34}(2^2-1)(2^4-1)(2^2-1) \\ &+ l_1 . 2^{-34+3}(2^4-1)(2^6-1)(2^2-1) - l_2 2^{-34+8}(2^2-1)(2^6-1)(2^4-1) \\ &+ l_3 2^{-34+15}(2^2-1)(2^4-1)(2^6-1), \end{aligned}$$

and in terms of our notation,

$$L_3 a_1^3 a_2^2 a_3 = -l_0 . a_1^2 a_2 + l_1 . 2^{b_1} a_1 a_2 a_3 - l_2 . 2^{b_2} a_1 a_2 a_3 + l_3 2^{b_3} a_1^2 a_2$$

or

$$L_3 . a_1 a_2 a_3 = -l_0 + \frac{a_3}{a_1} 2^{b_1} . l_1 - \frac{a_3}{a_1} . 2^{b_2} . l_2 + 2^{b_3} . l_3.$$

To give some indication of the degree of approximation afforded by these results the formula for L_5 was used to find the length of the arc of a quadrant of a circle; or taking the radius of the circle to be unity we may approximate to the value of $\frac{\pi}{2}$, the chords l_0, l_1, l_2 ... becoming

$$2\sin\frac{\pi}{4},\ 2\sin\frac{\pi}{8},\ 2\sin\frac{\pi}{16} \ldots .$$

The formula is in this case:

$$3.15.63.255.1023.\frac{\pi}{2}$$
$$=2^{36}\sin\frac{\pi}{2^7}-341.2^{25}\sin\frac{\pi}{2^6}+5797.2^{16}\sin\frac{\pi}{2^5}-5797.2^{9}\sin\frac{\pi}{2^4}$$
$$+341.2^{4}\sin\frac{\pi}{2^3}-2\sin\frac{\pi}{2^2}.$$

The first four terms in the right-hand member of this equation were evaluated from Vlack's ten-figure table of logarithms, the values for the six terms being

$$\begin{aligned}&1{,}686{,}460{,}375{\cdot}5-561{,}435{,}335{\cdot}5\\&\quad 37{,}237{,}906{\cdot}7-\quad 579{,}040{\cdot}5\\&\qquad 2{,}087{\cdot}9-\qquad\quad 1{\cdot}4\\&=1{,}723{,}700{,}370{\cdot}1-562{,}014{,}377{\cdot}4.\end{aligned}$$

The value of π may then be immediately expressed as

$$=2\times\frac{1{,}161{,}685{,}992{\cdot}7}{3\times15\times63\times255\times1023}$$
$$=\frac{1{,}161{,}685{,}992{\cdot}7}{369{,}776{,}637{\cdot}5}$$

or
$$\pi=3{\cdot}141{,}592{,}646,$$

a result which differs in defect of the true value 3·1415926535 by about 1 in 400,000,000.

R. M. MILNE.

MATHEMATICAL NOTE.

127. [E. 1. a.] *Elementary Treatment for Gamma Functions.**

Let
$$G(n,\,1+x)\equiv n^x/\{\prod_1^n(1+x/n)\},$$

and
$$\Gamma(1+x)=G(\infty,\,1+x);$$

changing $1+x$ into x, $G(n,\,x)=n^{x-1}/\{\prod_1^n(1+\overline{x-1}/n)\}$.

Thus
$$G(n,\,1+x):\,xG(n,\,x)=n:n+x,$$

and making $n=\infty$,
$$\Gamma(1+x)=x\Gamma(x). \qquad\text{.....(i)}$$

If x is a positive integer,
$$\Gamma(1+x)=\underline{|x}. \qquad\text{.....(ii)}$$

By definition, putting $x=1$,
$$G(n,\,2)=n/\{\prod_1^n(n+1)/n\}=n/(n+1);$$

so when $n=\infty$,
$$\Gamma(2)=1. \qquad\text{.....(iii)}$$

Also by (i),
$$\Gamma(1)=1,\quad \Gamma(0)=\infty. \qquad\text{.....(iv)}$$

Again,
$$G(n,\,1+x).\,G(n,\,1-x)=\prod_1^n\left(1-\frac{x^2}{n^2}\right)^{-1};$$

therefore when $x=\infty$,
$$\Gamma(1+x).\,\Gamma(1-x)=x\pi/\sin x\pi. \qquad\text{.....(v)}$$

For
$$x=\frac{1}{2},\quad \Gamma\left(\frac{3}{2}\right).\,\Gamma\left(\frac{1}{2}\right)=\frac{\pi}{2},\quad\text{and}\quad \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right);$$
$$\therefore\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}. \qquad\text{.....(vi)}$$

* Cf. *Math. Gaz.*, 1902, p. 160.

As an example of the Gamma function, let us consider

$$S_n=\Pi\left(1\pm\frac{x}{n}\right)\equiv(1+x)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\ldots\left(1\pm\frac{x}{n}\right),$$

when $x=\infty$.

By definition,

$$G\left(n,1+\frac{x}{2}\right):G\left(n,1-\frac{x}{2}\right)=n^x\prod_1^n\overline{2n-x}:\prod_1^n\overline{2n+x}.$$

Also $$S_{2n}\,.\,G(2n,1+x)=(2n)^x\prod_1^n\left(1-\frac{x}{2n}\right)\Big/\prod_1^n\left(1+\frac{x}{2n}\right);$$

$$\therefore\ 2^x\,G\left(n,1+\frac{x}{2}\right)\Big/G\left(n,1-\frac{x}{2}\right)=S_{2n}\,.\,G(2n,1+x);$$

$$\therefore\ \text{for } n=\infty,\quad S_\infty=2^x\,\Gamma\left(1+\frac{x}{2}\right)\Big/\Gamma\left(1-\frac{x}{2}\right).\,\Gamma(1+x)$$

$$=2^x/x\beta\left(x,1-\frac{x}{2}\right).$$

G. OSBORN.

REVIEWS.

Theoretische Arithmetik. (II. Abtheilung). By O. STOLZ und J. A. GMEINER. Leipzig, Teubner. 1902. Pp. 402.

This book is a new enlarged edition of certain chapters of Dr. Stolz' well-known *Allgemeine Arithmetik*. The merits of the latter are universally recognised, and no praise could be higher than to say that in lucidity and thoroughness the present volume is an improvement on it.

The first two chapters are not essential to the general plan of the book, the systematic development of the arithmetic theory of number, real and complex; but their inclusion is fully justified by their intrinsic interest. The first (chapter V. of the complete work) deals with the mathematical theory of *magnitudes*, and has no direct connection with number. Mr. Russell has shown in his recent work *The Principles of Mathematics* that the simplicity or complexity of a mathematical theory of magnitudes depends very much on the philosophical view which underlies it. The authors do not explicitly raise any philosophical questions, but what Mr. Russell calls the relative theory of magnitude is presupposed in their treatment of the subject. The result is an exceedingly complicated system of axioms. The working out of the theory on this basis, however, leaves nothing to be desired. The second chapter contains a very clear account of Euclid's theory of ratio. It is not a little remarkable that the part of Euclid's work which is perhaps least open to criticism is the part which most of us were brought up to despise.

The theory of real numbers is then developed from Cantor's point of view. A noteworthy feature of the authors' treatment of this part of the subject is that the fundamental theorems concerning limits are proved first for the particular cases in which the limits are rational numbers. This is much to be commended, although it adds considerably to the length of the chapter. Another feature of the book is the prominence given here and elsewhere to the theory of approximation by decimals. It is a pity that the authors have not seen their way to include at any rate an outline of the elements of the theory of aggregates and transfinite cardinals and ordinals. This would have added considerably to the value of the book.

The next two chapters deal with the elementary theory of powers, roots, logarithms, and series. These chapters do not present many very novel features, but are admirably thorough. I may perhaps mention particularly the sections on approximation to roots, on the solution of the functional equation

$$f(x+y)=f(x)\,f(y)$$

and on the development of the theory of logarithms from the equation

$$\log a=\underset{n=\infty}{L}\,n(\sqrt[n]{a}-1).$$

It is characteristic of the spirit of the book that a proof should be given that if $a_n = b_n$ $(n = 0, 1, 2, \ldots)$ the series Σa_n, Σb_n are *gleichartig*, *i.e.* converge, diverge, or oscillate together. Perhaps some mention of the theory of divergent series might be given: it is a little old-fashioned to say that the equation

$$1 - 1 + 1 - \ldots = \tfrac{1}{2}$$

has 'no meaning,' although, as the authors mean it, the remark is of course quite true. It is inevitable that in a book of this character most of the most interesting theorems about non-absolutely convergent series should be omitted, or relegated to the examples. And I may mention here that the examples (it is refreshing to see examples in a German book) are extremely well selected; they contain a large number of important theorems for which there is no space in the text, and nothing which is not of some interest, all 'tricks' being rigidly excluded. For instance, the development of the theory of e^x *from* its expression as a power series is given as an example. It is perhaps doubtful whether it is best to make the theory of e^x and $\log x$ entirely prior to the theory of series.

Of the remaining chapters only X. calls for special notice. Here the different possible varieties of complex numbers are considered, and it is shown that the ordinary complex numbers alone obey all the ordinary rules of arithmetic, and (after Frobenius) that only quaternions obey all but the commutative rule of multiplication.

Chapter XI. deals with the geometric theory of complex numbers; some very interesting geometric problems are solved as examples. The last chapters contain the theory of complex powers, roots, logarithms, and series.

The great merit of the book lies in the complete absence of sketchiness from any part of it; in this it affords a welcome contrast to many books written now. It is also unusually free from printers' errors. G. H. HARDY.

The Elementary Geometry of Conics. By the Rev. C. TAYLOR, D.D., LL.D., Master of St. John's College, Cambridge. Eighth edition, revised, pp. viii. 159. 4s. 6d. 1903. (Deighton, Bell.)

The new edition of this well-known text-book has been rewritten throughout, although the main lines of the sixth and seventh editions have been adhered to. Two chapters on the Central Conics and the Asymptotes have been replaced by chapters on the Ellipse and the Hyperbola respectively, and an interesting chapter on Newton's *Inventio Orbium*, with modifications, has been added. Some of the proofs have been simplified, such as the proofs that the pedal and orthoptic loci are circles, and that the ellipse and hyperbola are bifocal and symmetric with respect to the conjugate axis. The author has drawn on his stores of information for a few historical notes. These form a unique and valuable feature of the book, and might with advantage have been extended. The figures are for the most part accurate; and there is a wealth of examples, both in the text and at the ends of the chapters.

Notwithstanding its many merits, the book, regarded as a school text-book, has several demerits. Clearness is often sacrificed to conciseness of demonstration. The principle of passing from the general to the particular is too uniformly carried out, not only in placing the General Conic first, but in the chapters on the Conics separately. The book has a cramped and unattractive appearance. Some of the enunciations are indefinite, such as "A chord of a parabola being divided at any point, to determine the rectangle of its segments," and, "The two branches of a hyperbola make one curve." Cross-ratio is mentioned two or three times, but not defined. The explanation of plus and minus (p. 33) is inadequate and confusing, and all reference to the principle of continuity is deliberately omitted. The binding is so bad that the copy sent us is already falling to pieces.

The book is in many respects the best and most reliable text-book on the Geometry of Conics that can be obtained. F. S. MACAULAY.

Introduction to the Theory of Algebraic Equations. By L. E. DICKSON, Ph.D. Pp. v. 104. (Wiley & Sons, New York. 1903.)

A chapter on Substitutions and Resolvents has only recently been welcomed as an addition to Burnside and Panton's *Theory of Equations*. We now have in English a small self-contained book on the subject. It will prove very useful. There is just enough in it—exactly what is necessary and sufficient for the

purpose of exhibiting as simply as possible the facts as to why some equations can and others cannot be solved by radicals, with all interesting superfluity abstained from, or reserved for brief mention after business done. On matters like this it is harder to avoid bewildering the new student than to provide enough for his instruction; and Dr. Dickson has wisely preferred conciseness to elaboration.

After an introductory chapter on the General Quadratic, Cubic, and Quartic, in which the fact is enforced that the radicals which present themselves in the solutions are equal to rational functions of the roots, first notions as to substitutions and substitution-groups are explained. The general equation of any degree is then discussed from the group standpoint. The importance of a series of composition of the symmetric group of n letters is adequately indicated, and the factors of composition obtained. It is thus well brought home to the student that solution of a general equation of degree greater than 4 by means of a chain of binomial equations cannot be effected. Abel's completion of Lagrange's theory by proof that there can be no other means of algebraical solution is mentioned at this stage of the book, but his theorem is left to be established later on by Galoisian reasoning.

Here a first part of the book ends, and a new start is made. Our author's plan is to follow history in the sequence of ideas; and his separation of the book marks a strange historic discontinuity—an interval of the most painful human interest. The interval separates the completion of the older Lagrangian research as to *general* equations and the tardy discovery by mathematicians that a new and more profound theory, dealing alike with *general* equations and *special*, had been bequeathed to them by Evariste Galois—a perverse genius of a boy, who had been long forgotten, but whose death in a duel in 1832, while he was still not of age, had seemed at the time to end a short story of utter failure.

There is more than historic fitness in the division of the book, small though it is, into two parts. Many will read the first with benefit at an earlier stage of mathematical development than one who is prepared to master the second part must have reached. The Lagrangian theory, complete in itself, has in the first part been placed, but for its last word, well within the grasp of the ordinary intelligent student. But the theory of Galois can never be made easy. It demands prolonged retention of a delicate thread of reasoning, while novel subsidiary ideas of provoking fugitiveness are in succession grasped and not let go. Though no analysis beyond that used by the schoolboy is wanted, there is requirement either of mature judgment or of quite exceptional powers of concentration and assimilation. Though expressed in its lowest terms, a work on the subject must be regarded as written for the serious mathematician rather than the junior student.

The marshalling of essentials in the second part of Dr. Dickson's book is very skilful, and the trained mathematical enquirer into the scope and beauty of the Galoisian theory of resolvents, in domains of rationality widened by successive adjunctions of irrationals, will be satisfied by it on the main thesis, and incited to enquire further elsewhere. A very good instance of the clearness with which the author succeeds in introducing requisite ideas occurs in chapter vii. where the notion of a quotient group G/H has to be brought in. Had readers been expected to be novices in mathematics, rather less brevity would have been desirable in a few places, *e.g.* in enforcing the corollary to § 58, in making evident the bearing of all the conditions as to ψ in the lemma of § 65, and in proving § 75.

The book is quite free from any misprint which could mislead. It is an odd volume to read and handle—closely printed, in rather thin type, on abnormally thick heavy paper.

Readers of the *Mathematical Gazette* are keenly interested in questions of mathematical education, and they may desire an opinion as to whether the belated appearance in English of a new kind of text-book on Theory of Equations, with all emphasis on the word *Theory*, ought to lead to any great change, in the teaching of theory of Equations, as we have understood it, to ordinary classes. One thinks not. The knowledge of relations between roots and coefficients, of elementary facts as to the transformation of equations, of the simpler notions as to existence, reality, equality, and rationality of roots, is knowledge which helps to the intelligent employment of methods of exact and

approximate numerical solution, and as such should continue to be imparted at an early stage to all classes of mathematical students—to the utilitarian as well as to the future follower of mathematics whithersoever it may lead. Such practical information cannot be interwoven with the exposition of subtle logical theories with advantage either to the beginner or to the specialist. The interests of all will be best served by letting what is meant for the latter continue to appear in second volumes or separate treatises. E. B. ELLIOTT.

Experimental and Theoretical Geometry. By A. T. WARREN, M.A. (Clarendon Press). **Elementary Geometry.** Pts. I. and II. By F. R. BARRELL, M.A., B.Sc. (Longmans, Green & Co.). **A School Geometry.** Pts. I. and II. By H. S. HALL, M.A., and F. H. STEVENS, M.A. (Macmillan & Co.).

These treatises all aim at providing a course of Geometry suitable for replacing Euclid's Elements as a text-book. They follow the recommendations of the Mathematical Association in treating angles before triangles, and in using hypothetical constructions. The course seems in each case well chosen and plentifully furnished with varied collections of riders, as well as with exercises in Geometrical Drawing and numerical calculation. Mr. Warren, as he indicates in his title, gives an experimental course, which occupies nearly half the book. This, though not so extensive as Mr. Eggar's Practical Exercises, is of much the same general character, and forms a valuable addition to the school library.

We feel sure that those who use it will find it suggestive and stimulating. We welcome the short chapter on the construction of certain solids, being convinced that to show boys and girls how 'rule and compass' work on the flat helps them to plan out and carry through the construction of various models (trays, boxes, etc.), is one of the surest ways of awakening and keeping awake their interest in Geometry. It will be an advantage to some teachers to have their experimental and theoretical courses in the same book. In the remaining 135 pages the author gets through the ordinary school course of plane Geometry.

Mr. Barrell's Part I. (Subject matter of Euclid I.), and Part II. (Subject matter of Euclid's III., 1-34, IV. 4-9), form Section I. of a work which is to be completed by the addition of two other sections; of which one more will be devoted to plane geometry, and the other will provide a course of Solid.

There is a useful introduction of 20 pages. The course is illustrated by a large number of diagrams, in which effective use is made of shading. Angles are, as a rule, denoted by single Greek letters, whose pronunciation is carefully given.

In the remaining pages Mr. Warren and Mr. Barrell both obtain the value of the sum of the interior angles of a triangle by the rotation proof, and are thus enabled to place Euclid's I. 32 early in the course. Excellent as this method is as an illustration, we have some doubts as to whether it is readily appreciated by beginners.

Messrs. Hall and Stevens also issue their Parts I. and II. as instalments of a complete School Geometry. We do not doubt from an inspection of the parts already issued that it will be widely used, especially among those teachers who intend to adhere to Euclid's rigour of treatment, while getting rid of the tiresome disfigurements caused by some of his self-imposed restrictions.

Elementary Geometry. By CHINTAMANI MUKERJI, B.A. Pp. xvi., 124. 1903. (Allahabad Indian Press).

A useful little treatise dealing with the subject matter of Euclid's I. and II. In Book I. the author proceeds on much the same lines as the English writers, whose works have been already noticed, except that he uses fewer symbols. In Book II. he retains the geometrical treatment. A useful index of mathematical terms is supplied. We notice that the author has followed Mr. Croome Smith speaking of "the parallelograms about the diameter of a parallelogram" as its *primaries*, and their complements as its *secondaries*. These seem useful abbreviations. The price is 10 *annas*.

Euclid. Bks. V., VI., XI. By R. DEAKIN, M.A. Pp. 144. 1903. (University Tutorial Press.)

This completes the edition of the Elements published as part of the

'Tutorial' series. It is marked by the same features as the section containing Bks. I.-IV. It contains useful sets of exercises, hints to students, and summaries of results.

Solid Geometry. By DR. F. HOCEVAR. Translated and adapted by C. Godfrey, M.A., and E. A. Price, B.A. Pp. vi., 80. 1903. (A. & C. Black.)

We congratulate the adapters on the opportune appearance of this excellent treatise, for which we have nothing but praise. It contains a systematic and adequate treatment of the Solid Angle, Prism, Cylinder, Pyramid, Cone, Sphere, and Regular Polyhedra. The design and execution of the diagrams are quite equal to those of the American treatises which we have previously had occasion to single out for praise in this respect. Excellent sets of practical and suggestive exercises are given. Some terms which, though not altogether new, have been unusual in text-books, are advantageously introduced, *e.g.* 'skew,' 'net,' 'cap,' and Mr. Hayward's 'cuboid.' Answers to the numerical exercises and a comprehensive index are also provided.

Practical Plane and Solid Geometry. By J. HARRISON, M.I.M.E. Pp. xiii., 250. (Macmillan & Co.)

This seems to deserve thoroughly the recommendation given to it by Professor Perry in an introductory note. The author seems to have realised his aim—to present and develop the subject-matter in such a way that the reader not only acquires a knowledge of geometrical principles, but is trained so as to be able to put his knowledge to practical use. We find our own high opinion of it confirmed by one experienced in the practical applications of the principles explained.

E. M. LANGLEY.

Leçons Elémentaires sur la théorie des Fonctions Analytiques. Première partie. Chapitres I. à V. By EDOUARD A. FOUET. Pp. 330. (Gautier-Villars.)

The general idea of this work, as far as can be gathered from the five chapters before us, is the development of the theory of Functions on the lines of both Cauchy and Weierstrass. The author's fundamental position is "the idea of number and the notion of function are at the basis of Analysis." Starting from this position a detailed introduction is devoted to the discussion and explanation of the ideas of numbers and of function. This introduction is furnished with very full references, but in spite of this there is a suggestion of superficiality in the treatment of aggregate theory, and we think that many of the footnotes might have been with advantage incorporated in the text. The part of the introduction concerned with the general concept of function, on the other hand, is developed in a very pleasing manner. The exposition of the idea of continuity and of the nature of analytic functions also leaves little to be desired. Chapter I. is concerned with the general properties of analytic functions, and contains an interesting section on algebraic functions. In the two succeeding chapters the general properties of series are discussed. Particular mention may be made of the section on trigonometric series, and of that on the series connected with elliptic functions, both of which are written in very attractive style.

The remainder of the book consists of two chapters, one on the integral theory as developed by Cauchy, the other being an *exposé* of the subject from the Weierstrassian point of view.

On the whole, the book, though parts of it are written in pleasing style, appears to contain very little new information, and does not seem to supply any want.

J. E. WRIGHT.

The Slide Rule. By R. G. BLAINE. 2nd edition. Pp. xii., 152. 1903. (E. & F. Spon.)

This little treatise forms an excellent introduction for the student who is endeavouring to master the intricacies of the Slide Rule. Great stress is laid on contracted methods of calculation at the outset, before proceeding to explain the theory of logarithms or the use of the rule itself. Not the least attractive feature to the student is the collection of examples illustrating the various applications of an instrument which is coming more and more into general use.

A Celluloid Slide Rule, with spare Logologarithmic Slide. 10 in., 15s.; with case, 16s. 6d. Arranged by Lt.-Col. H. C. DUNLOP, R.F.A., and C. S. JACKSON, M.A. (sold by J. Davis & Son, All Saints' Works, Derby.)

The principle of the log. log scale was described as long ago as 1814 in the *Philosophical Transactions*, but the question of expense has prevented its production in a cheap enough form to become of general practical utility. The paper read at the last general meeting, by Mr. C. S. Jackson, and to be hereafter published in the *Gazette*, will shew in detail the advantages presented by the addition of the log. log scale to the ordinary slide rule. The rule sent to us by Messrs. Davis is very carefully engraved, and is well worth the price charged for it. But we should like to see on the market a slide rule within the reach of boys at school. There seems to be no reason why such an instrument, say a ten-inch rule correct to one part in 200 or 300, should not be offered to the enormous public represented by our secondary schools for anything between a shilling and half-a-crown.

Beginners' Algebra. By M. S. DAVID. Pp. viii., 232. (A. & C. Black.)

This is an eminently sensible little introduction to Algebra for little boys. It is written, in the main, on the lines of the suggestions of the Committee of the Mathematical Association, although "outlined and partly written" before the report of the Committee was published. The use of squared paper is introduced as soon as the pupil understands the meaning of an equation. Detached coefficients are introduced with multiplication. Homogeneity, symmetry, and form are not forgotten. The answers are published separately.

Elements of Finite Differences. By J. BURN and E. H. BROWN. Pp. 259. 1902. 7s. 6d. net. (C. & E. Layton.)

This book is prepared for the benefit of students who propose to enter for Part I. of the Examinations of the Institute of Actuaries. Hitherto, we imagine, the pupil has been compelled to select what seemed advisable from the pages of Boole; from an ill-digested study of that famous volume he is now spared by the labours of our authors. They have succeeded in giving a very clear and succinct account of those portions of the subject which are necessary for the purposes of this particular examination, and as an instance of the excellence of their exposition we would commend the way in which, in connection with Lagrange's theorem, the theory of interpolation is presented. The authors do not follow blindly the methods of Boole. For instance, Δx is allowed to assume any arbitrary value, and E is defined as $Eu_x = u_{x+h}$ instead of $Eu_x = u_{x+1}$. The rest of the volume consists of the questions set in recent years at the examinations, with solutions.

Elements of Plane Analytical Geometry. By G. R. BRIGGS and MAXIME BÔCHER. 7th edition. Pp. vi., 191. 1903. (Wiley, Chapman, Hall.)

This book has been used by the freshman's classes at Harvard for the last twenty years or more, and forms an adequate first course in analytical geometry. The present edition contains additions from the pen of Professor Maxime Bôcher. The whole is well and carefully written, and is adapted for the purpose for which it is intended. We notice that, in the equation $y = mx$, m is called the *slope* of the line.

Éloges Académiques. Nouvelle Série. By JOSEPH BERTRAND. Pp. 400. 3 fr. 50c. 1902. (Hachette.)

This is a collection of the orations pronounced by the late J. Bertrand on various occasions, in his official capacity as perpetual secretary to the Académie des Sciences. The names of those whose "éloges" are included in this volume are: Poinsot, Cosson, Chasles, Cordier, Tisserand, Viète, Galileo, Paris, Cauchy, Papin, Clairaut, Euler, d'Alembert, Lagrange, Abel, Galois, Faraday, and Pasteur. Space forbids our doing more than to draw attention to the collection, which also contains an "éloge historique" on Bertrand himself by no less an authority than M. Gaston Darboux. There are many purple patches throughout the speeches, and as models of this type of oratory we may refer the reader to those which deal with the life and works of Poinsot, Abel, and Galois.

Plane and Spherical Trigonometry. By G. A. WENTWORTH. 2nd edition. Pp. vii., 207, 25. (Ginn & Co., St. Martin's Street, W.C.)

This is a simply written and admirably illustrated introduction to the elements of Plane and Spherical Trigonometry. The proofs are simple and concise, and show evidence of considerable thought and attention. All but the absolute essentials of the subjects are omitted.

Niedere Zahlentheorie. By P. BACHMANN. Pp. x., 402. 1902. (Teubner.)

The publishers of this volume of the well-known *Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften* could scarcely have selected a more capable writer on the elementary theory of numbers than Herr Bachmann. The volume in no way clashes with what has already been written on the subject by the author, to which indeed he considers it may be taken as a supplementary volume. After a preliminary historical disquisition, the author enters upon a discussion of integers, based on the idea of manifolds and cuts. The distinction between finite and infinite *ensembles* is clearly exhibited, and the finite nature of the natural number is displayed. The theory of the fundamental operations, and the introduction of the negative numbers and zero conclude the first chapter. The fundamental theorems of divisibility are based on the notion of the modulus. An excellent section is that dealing with the highest power of prime numbers occurring in a factorial. From congruences and the properties of the function $\phi(n)$ the author proceeds to a detailed examination of the Euclidean and other algorithms; Farey's series and the series deduced therefrom for the purpose of approximations to a rational or irrational number w—series called by Christoffel and Hurwitz the *characteristics* of the number w, (*Ann. di Matt.*, 1888); Stern's series :—

$$r,\ r+s,\ s,$$
$$r,\ 2r+s,\ r+s,\ r+2s,\ s,$$
$$r,\ 3r+s,\ 2r+s,\ 3r+2s,\ r+s,\ 2r+3s,\ r+2s,\ r+3s,\ s.$$

where each row is obtained by intercalating the sum of two consecutive terms; and lastly to a numerical function connected with these series and utilised by Eisenstein in his investigations into the generality of the law of reciprocity, (*Journ. f. Math.*, 39, 1850; *Berl. Monats.* 1850). The next chapter deals with the theorems of Fermat and Wilson, at which we may be said to come to an end of the subject as taught in schools. We now arrive at the gem of the book, an exhaustive treatment of the theory of quadratic residues and of the law of reciprocity. The author gives a detailed study of quite a couple of score of proofs of this law, making the chapter almost a complete up-to-date monograph. The last chapter treats of the higher congruences, concluding with Galois' imaginaries and the 7th Gaussian proof of the law so fully treated in the preceding chapter. We need hardly add that readers of this volume will look forward with much eagerness to that which is yet to come.

Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. By Dr. K. GEISSLER. Pp. viii., 417. 1902. Teubner.

We received this book with the most pleasurable anticipations. There is hardly any feature in the trend of thought during the last thirty or forty years which is more striking than the critical investigations into the nature of the principles of Mathematics. The assumptions and the very concepts underlying those principles have been seriously challenged. Their philosophical basis is still engaging the attention of some of the acutest minds of our time. One naturally expected—one might be pardoned for expecting—from a countryman of Cantor, to whom we practically owe the theory of mathematical infinity, a treatise which would, in the light of modern research, assist us to a grasp of the fundamental verities, even if the disciple displayed in a less degree the limpid lucidity of his Master. It is but rarely, too, that an ambitious work dealing with subjects of the first importance is issued by the premier publishing firm in Germany without justifying in a high degree the wisdom of their decision. We are most unwillingly compelled to state that our bright anticipations are not realised by a

perusal of the volume under notice. Although the author is by no means without ability, he does not seem to us to possess the competence, *quâ* mathematician or philosopher, which alone would justify his undertaking a task of such difficulty as a presentment of the doctrine of infinity. As a mathematician he should be thoroughly acquainted with the labours of Cantor. He should be able to wield with effect that most potent of modern weapons of investigation—the Symbolic Logic of Peano. These qualifications are the irreducible minimum with which a writer should be furnished, should he wish to approach his subject from the mathematical side. Further, he should be a metaphysician, a logician, and perhaps it would be no disadvantage to be a psychologist. Now these twofold desiderata are, to say the least of it, somewhat infrequently met with in one and the same individual, and one would instinctively look to Germany for the man of such attainments. As Dr. Hobson has pointed out in the valuable sketch of the analytical theory of infinity which formed his Presidential address to the London Mathematical Society last November: "The feeling of impatience with which men regard a hypercritical attitude towards the fundamental concepts of Mathematics exists perhaps in exceptionally large measure in the English mind, whose genius is in a preponderating degree directed towards the concrete, and upon which to a considerable extent peculiarly abstract questions seem to exercise a peculiar repulsion." Fortunately, however, we are able to point to a conspicuous instance of an Englishman who possesses in quite a remarkable degree the necessary equipment. Few of our readers who have seen Mr. Bertrand Russell's first volume of the *Principles of Mathematics*, especially Part V., can feel any hesitation as to whom we allude. The section which deals with the problem of infinity, some one hundred pages, has just those qualities of precision and consistency which a treatise like this of Dr. Geissler seems to lack. The writer is willing to admit that he has found it difficult at times to follow Dr. Geissler throughout his arguments, which are almost inexplicable where preliminary assumptions are over-clouded by intuitional conclusions. The book is divided into two parts, the first of which is devoted to the consideration of a series of questions, mainly from Geometry and Dynamics, in which the idea of infinity is involved. In modern treatises most of these questions are dealt with in a perfectly satisfactory manner by the aid of accurate definitions, and they present no real difficulties from the purely mathematical point of view. We cannot help feeling that it would have been far better if the author had confined his attention to those arithmetical problems into which the doctrine of continuity enters—Cantor's work on transfinite numbers, for instance—for it is from the notion of an arithmetical continuum that the treatment of infinity must start. However that may be, the author has chosen otherwise, and has deliberately selected a class of problem which is nowadays solved by the applications of the doctrine of limits, and not by the doctrine of infinitesimals at all. He uses the antiquated orders of the infinite and the infinitesimal, and bolsters up his position, the insecurity of which apparently he has detected, with the aid of a brand-new doctrine of his own—a theory partly logical and partly metaphysical, to which he has given the name of *Behaftungen*. His arguments seem to be vitiated throughout by a certain obscurity of view as to the nature of continuity. We may almost apply to him what Mr. Russell so neatly says of the attitude of the philosophers to that notion: "As to what they meant by continuity and discreteness, they preserved a discreet and continuous silence; only one thing was evident, that whatever they did mean could not be relevant to mathematics, or to the philosophy of space and time." We do not accept the validity of the intrusion of geometry into the general argument. There is too much that is yet obscure in the exact nature of geometrical continuity, and there is at present but little reason to suppose that it is legitimate to call upon the resources of arithmetic to assist us in presenting a clear outline amid that obscurity. In the second part of his volume, the author seems to lead us even further astray, by appealing to the psychological origin of our ideas. This would appear to make confusion worse confounded. We cannot admit that the senses play any necessary part of our conceptions of the finite, the infinite, or the infinitesimal. It is because half the arguments in the book are based upon intuitional conclusions that we consider Dr. Geissler has failed to throw any real light on the problem which he has so cheerfully attacked. We are therefore compelled to say that his *Grundsätze und das Wesen des Unendlichen* can only be described as a *coup manqué*.

PROBLEMS.

491. [L¹. 10. b.] The sides of a triangle LMN touch a parabola at the points P, Q, R, the normals at which meet in a given point O. Find the equation of the radical axis of the circles LMN, PQR, and shew that if O lie on a fixed straight line, the radical axis passes through a fixed point.

A. F. VAN DER HEYDEN.

492. [L¹. 11. c.] The locus of the centre of an equilateral hyperbola which has contact of the second order with the logarithmic curve $y=m\log x$, and whose asymptotes are parallel to the coordinate axes, is an equal logarithmic curve.

V. RETALI.

493. [L¹. 7. d.] The directrices of the conic given by the general equation

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

are given by $$\sqrt{R+a-b}\,\frac{\partial S}{\partial x}+\sqrt{R+b-a}\,\frac{\partial S}{\partial y}\pm 2\sqrt{2\Delta}=0,$$

where $$R^2=(a-b)^2+4h^2.$$ (C.)

494. [A. 2. a.] If the equations

$$\begin{aligned} axy+bx+cy+d&=0,\\ ayz+by+cz+d&=0,\\ azw+bz+cw+d&=0,\\ awx+bw+cx+d&=0 \end{aligned}$$

are satisfied by values of x, y, z, w, which are all different, then

$$b^2+c^2+2ad.$$ (C.)

495. [R. 1. a.] A coastguardsman, walking steadily along a straight footpath at the top of a level cliff, sees a steamer at sea in a direction at right angles to the path; at the ends of successive intervals, each of 15 minutes, the directions in which the steamer is seen make angles of 60° and 30° respectively with the path: assuming that her course is straight, prove that she will disappear round the headland at the end of the path after another 30 minutes. (C.)

496. [R. 7. b. γ.] Two planes are inclined at angles θ, ϕ to the horizon and the horizontal lines in them include an angle ψ; two particles are projected on the planes with an initial relative velocity parallel to a certain plane: shew that the relative path is a parabola in a vertical plane, and find the inclination of the axis to the vertical. (C.)

497. [R. 4. c.] Two uniform rods AB, AC, of weights W, W', are rigidly joined at A, and AB is placed on a rough peg: shew that there is, in general, a length $> \frac{\mu W' p}{W+W'}$ on AB, such that if the peg touch AB anywhere within this space the system will be in equilibrium, p being the length of the perpendicular from C on AB and μ the coefficient of friction. (C.)

498. [R. 4. a. α.] A triangular plate is held in equilibrium by three given vertical strings of equal lengths fastened to the vertices and to three fixed points in the ceiling. What couple will keep the plate in equilibrium if turned through a given angle θ about the vertical through its centre of gravity? (C.)

499. [A. 2. a.] Solve for x, y, ξ, η the equations

$$\xi+\eta = a,$$
$$\xi x+\eta y = b,$$
$$\xi x^2+\eta y^2 = c,$$
$$\xi x^3+\eta y^3 = d.$$

Discuss the case in which a, b, c, d are in G.P. Melbourne, 1898.

500. [R. 4. c.] A weight W is carried by a pin at C connecting the two equal bars AC, BC, each of weight w, whose ends A, B are hinged at two fixed points of a vertical wall, A being vertically above B: find the reactions at the three joints. Melbourne, 1897.

SOLUTIONS.

418. [R. 7. f.] *A circular wire of radius a moves in its own plane without rotation, so that its centre has a simple harmonic motion of amplitude a; a bead moves on the wire uniformly, completing a circuit in the period of the simple harmonic motion, and being in the line of the motion of the centre when the centre is in its mean position: prove that the acceleration of the bead is towards the centre of the simple harmonic motion, and that its path is an ellipse of eccentricity* $(\frac{3}{2}\sqrt{5}-\frac{5}{2})^{\frac{1}{2}}$. Pemb. (C.), 1897.

Solution by W. F. Beard: C. V. Durell.

Let O be the mean position of the centre: take axes Ox in the direction in which the centre is moving, Oy in the direction in which the bead is moving. After a time t let A be the position of the centre, and P that of the bead. The acceleration of P is compounded of $\omega^2 PA$ towards A and $\omega^2 AO \parallel$ to AO. Its resultant acceleration is therefore $\omega^2 PO$ along PO. The coordinates of P are $(a\sin\omega t \pm a\cos\omega t,\ a\sin\omega t)$: its path is therefore

$$y^2+(y-x)^2 \pm \omega^2,$$

an ellipse of eccentricity $(\frac{3}{2}\sqrt{5}-\frac{5}{2})^{\frac{1}{2}}$.

421. [D. 2. d.] (1) *If $\frac{p_n}{q_n}$ be the n^{th} convergent of* $a-\dfrac{1}{a-}\ \dfrac{1}{a-}\ \ldots$, *shew that*

$$p_n^2-ap_nq_n+q_n^2=p_n^2-p_{n+1}p_{n-1}=1,$$

and find the value of

$$a^2(p_nq_{n-1}+q_np_{n-1})-2a(p_np_{n-1}+q_nq_{n-1}).$$

(2) *If $\frac{\pi}{2}, \alpha_1, \alpha_2 \ldots \alpha_n$, be a series of angles such that*

$$\cot\alpha_2=\mu\cot\alpha_1-\cot\frac{\pi}{2},$$

and, generally, $\cot\alpha_n=\mu\cot\alpha_{n-1}-\cot\alpha_{n-2}$ *where* $\mu=2\operatorname{cosec}\alpha_1$;

shew also that $\operatorname{cosec}\alpha_2=\mu\operatorname{cosec}\alpha_1-\operatorname{cosec}\frac{\pi}{2}$,

and, generally, $\operatorname{cosec}\alpha_n=\mu\operatorname{cosec}\alpha_{n-1}-\operatorname{cosec}\alpha_{n-2}$. E. Budden.

(1) *Solution by* Proposer, W. F. Beard *and* J. F. Hudson.

The ordinary law of convergents gives

$$p_n=ap_{n-1}-p_{n-2}; \quad \ldots\ldots(1)$$

also since $\dfrac{p_n}{q_n}=a-\dfrac{1}{\frac{p_{n-1}}{q_{n-1}}}$,

$$q_n=p_{n-1}; \quad \ldots\ldots(2)$$

$$\therefore\ p_n^2 - aq_np_n + q_n^2 = p_n^2 - p_{n-1}(ap_n - p_{n-1}),\ \text{by (2)}$$
$$= p_n^2 - p_{n-1}p_{n+1},\ \text{by (1)} \quad \ldots\ldots(3)$$
$$= p_nq_{n+1} - q_np_{n+1},\ \text{by (2)}$$
$$= 1. \quad \ldots\ldots(4)$$

Again the value of the given expression

$$= a^2(p_np_{n-2} + p^2_{n-1}) - 2ap_{n-1}(p_n + p_{n-2}),\ \text{by (2)}$$
$$= a^2(p_np_{n-2} + p^2_{n-1}) - 2a^2p^2_{n-1},\ \text{by (1)}$$
$$= -a^2(p^2_{n-1} - p_np_{n-2})$$
$$= -a^2,\ \text{by previous theorem.}$$

(2) *Solution by* W. F. Beard.

$$\cot a_2 - \mu \cot a_1 = 0,$$
$$\cot a_3 - \mu \cot a_2 + \cot a_1 = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\cot a_n - \mu \cot a_{n-1} + \cot a_{n-2} = 0.$$

From these equations we can easily find that $\cot a_1$, $\cot a_2$, etc., are coefficients of x, x^2, etc., in the expansion of $\dfrac{a_1x}{1-\mu x+x^2}$, where $a_1 = \cot a_1$.

Let $1 - \mu x + x^2 \equiv (1-\alpha x)(1-\beta x)$,

so that $\alpha\beta = 1,\ \alpha + \beta = \mu$; .....(i)

$$\therefore\ \frac{a_1x}{1-\mu x+x^2} = \frac{a_1x}{(1-\alpha x)(1-\beta x)} = \frac{a_1}{\alpha-\beta}\left\{\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right\} \quad \ldots\ldots(ii)$$

$$\alpha - \beta = \sqrt{\mu^2 - 4} = \sqrt{4(\operatorname{cosec}^2 a_1 - 1)} = 2\cot a_1 = 2a_1.$$

Hence $\cot a_n$ = coefficient of x^n in (ii) $= \dfrac{a_1}{2a_1}\cdot(\alpha^n - \beta^n) = \dfrac{\alpha^n - \beta^n}{2}$;

$$\therefore\ \operatorname{cosec}^2 a_n = 1 + \cot^2 a_n = \frac{(\alpha^n - \beta^n)^2}{4} + 1 = \frac{(\alpha^n + \beta^n)^2}{4};$$

$$\therefore\ \operatorname{cosec} a_n = \frac{\alpha^n + \beta^n}{2}$$ supposing the angles acute;

and this is true for $a_0 = \pi/2$.

Thus $\mu \operatorname{cosec} a_{n-1} - \operatorname{cosec} a_{n-2} = \frac{1}{2}\{\mu(\alpha^{n-1} + \beta^{n-1}) - \alpha^{n-2} - \beta^{n-2}\}$
$= \frac{1}{2}\{(\alpha+\beta)(\alpha^{n-1} + \beta^{n-1}) - \alpha\beta(\alpha^{n-2} + \beta^{n-2})\}$ from (i)
$= \frac{1}{2}(\alpha^n + \beta^n)$
$= \operatorname{cosec} a_n.$ Q.E.D.

Solution by A. W. Poole.

Let $u_n = \cot a_n$, etc., $u_0 = \cot a_0 = \cot \dfrac{\pi}{2} = 0$;

$$\therefore\ u_n - 2\operatorname{cosec} a_1 \cdot u_{n-1} + u_{n-2} = 0.$$

The roots of the equation $t^2 - 2\operatorname{cosec} a_1 \cdot t + 1 = 0$ are

$$t_1 = \cot\frac{a}{2};\quad t_2 = \tan\frac{a}{2};$$

$\therefore$ we have $u_{n+1} - (t_1 + t_2)u_n + t_1t_2u_{n-1} = 0$;

$$\therefore\ u_{n+1}-t_2u_n=t_1(u_n-t_2u_{n-1})$$
$$=t_1^2(u_{n-1}-t_2u_{n-2})$$
$$=\ldots\ldots\ldots\ldots\ldots$$
$$=t_1^n(u_1-t_2u_0).$$

Also $$u_{n+1}-t_1u_n=t_2^n(u_1-t_1u_0)\ ;$$
$$\therefore\ (t_1-t_2)u_n=(u_1-t_2u_0)t_1^n-(u_1-t_1u_0)t_2^n$$
$$=u_1(t_1^n-t_2^n),$$

and $$t_1-t_2=\cot\frac{a_1}{2}-\tan\frac{a_1}{2}=2\cot a_1=2u_1\ ;$$

$$\therefore\ 2u_n=\cot^n\frac{a_1}{2}-\tan^n\frac{a_1}{2}\ ;$$

$$\therefore\ \cot a_n=\frac{1}{2}\left[\cot^n\frac{a_1}{2}-\tan^n\frac{a_1}{2}\right]$$

Now $$\operatorname{cosec} a=\sqrt{1+\cot^2 a}\ ;$$
$$\therefore\ \operatorname{cosec} a_n=\left[1+\frac{1}{4}\left\{\cot^{2n}\frac{a_1}{2}-2+\tan^{2n}\frac{a_1}{2}\right\}\right]^{\frac{1}{2}}$$
$$=\frac{1}{2}\left(\cot^n\frac{a_1}{2}+\tan^n\frac{a_1}{2}\right),$$

and $\mu\operatorname{cosec} a_{n-1}-\operatorname{cosec} a_{n-2}$

$$=\operatorname{cosec} a_1\left(\cot^{n-1}\frac{a_1}{2}+\tan^{n-1}\frac{a_1}{2}\right)-\frac{\cot^{n-2}\frac{a_1}{2}+\tan^{n-2}\frac{a_1}{2}}{2}$$
$$=\frac{\left(\cot\frac{a_1}{2}+\tan\frac{a_1}{2}\right)}{2}\left(\cot^{n-1}\frac{a_1}{2}+\tan^{n-1}\frac{a_1}{2}\right)-\frac{\cot^{n-2}\frac{a_1}{2}+\tan^{n-2}\frac{a_1}{2}}{2}$$
$$=\frac{1}{2}\left(\cot^n\frac{a_1}{2}+\tan^n\frac{a_1}{2}\right)\ ;$$
$$\therefore\ \operatorname{cosec} a_n=\mu\operatorname{cosec} a_{n-1}-\operatorname{cosec} a_{n-2}.$$

Solution by PROPOSER.

Denoting $\cot a_n$ by u_n, $\operatorname{cosec} a_n$ by v_n, suppose the theorem true for all values of n up to n; and let $v=\mu v_n-v_{n-1}$. Then

$$v^2-u_{n+1}^2=\mu^2(v_n^2-u_n^2)+(v_{n-1}^2-u_{n-1}^2)-2\mu(v_nv_{n-1}-u_nu_{n-1}).$$

Now $$v_n^2-u_n^2=1=v_{n-1}^2-u_{n-1}^2\ ;$$

and if $$f_n=v_nv_{n-1}-u_nu_{n-1},$$

we can readily verify that

$$f_n+f_{n-1}=\mu=\ldots=f_3+f_2.$$

But $$f_2=\frac{\mu}{2}=f_3=f_n=\ldots=f_n\ ;$$

$$\therefore\ v^2-u_{n+1}^2=\mu^2+1-\mu^2=1.$$

Hence, remembering $u_{n+1}=\cot a_{n+1}$, and assuming $a_1, a_2, \ldots$ acute angles,

$$\operatorname{cosec} a_{n+1}=v=\mu v_n-v_{n-1}\,(\text{hyp.})=\mu\operatorname{cosec} a_n-\operatorname{cosec} a_{n-1}.$$

Hence, if the theorem is true up to a_n it is true for a_{n+1}.
But it is true for a_3, and therefore generally.

Note if $\frac{p_n}{q_n}$ be the nth convergent of $\mu-\cfrac{1}{\mu-}\cfrac{1}{\mu-}$, etc., it is readily seen that

$$\cot a_n=p_{n-1}\cot a_1,\ \text{ and }\ \operatorname{cosec} a_n=p_{n-1}\operatorname{cosec} a_1-p_{n-2}.$$

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie. Pa.)].

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. Complete. Vols. 1-35. Half leather. £25.

Also Vols. 25-35 of the same. Cloth. Half price, £7.

(2) **Wanted.**

Vols. I.-IV. *Mathésis.* 1881-1884.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

THE LIBRARY.

The Librarian begs to acknowledge, with thanks, the gift of the following book :

Greek Geometry, from Thales to Euclid. By GEORGE JOHNSTON ALLMAN, LL.D., D.Sc., F.R.S.

Presented by the Author.

BOOKS, ETC., RECEIVED.

The Elements of Finite Differences. By F. BURN and E. H. BROWN. pp. 260. 7s. 6d. 1902. (Layton.)

Theory of Observations. By T. N. THIELE. pp. iv., 144. 12s. net. 1903 (Layton.)

Elementary Geometry. By C. MUKERJI. pp. xvi., 124; n.p. 1903. (Indian Press, Allahabad.)

American Journal of Mathematics. Edited by FRANK MORLEY and others. Vol. xxv. No. 2. Ap. 1903.

The Double-Six Configuration connected with the Cubic Surface, and a related Group of Cremona Transformations. Kasner. *Untersuchungen über lineare Differentielgleichungen h-ordnung und die Zugehörigen Gruppen.* Epsteen. *The Logic of Relations, Logical Substitution Groups, and Cardinal Numbers.* A. N. Whitehead. *On Differential Equations belonging to a Ternary Linearoid Group.* Ross. *On a Certain Group of Isomorphisms.* Young.

Practical Plane and Solid Geometry. By J. HARRISON. pp. xiii., 250. New edition. 2s. 6d. 1903. (Macmillan.)

Elementary Geometry. Section I. Parts I. and II. By PROF. F. R. BARRELL. pp. xii., 168. 1s. each. 1903. (Longmans.)

Essai sur la Sommation de quelques Séries Trigonométriques. By E. ESTANAVE. pp. 112. 6 frs. 1903. (Hermann.)

Preliminary Geometry, Theoretical and Practical. By G. H. WYATT. pp. x., 82. 1903. 1s. 4d. (Relfe.)

Tests in Geometry (for Preliminary and Junior Candidates for Cambridge Locals.) By W. SLADE. pp. 26. 6d. each. 1903. (Relfe.)

Experimental and Theoretical Course of Geometry. By A. J. WARREN. pp. viii., 248. 1903. (Clarendon Press.)

The Open Court. May, 1903. 6d. (Kegan Paul.)

On the Determination of High Temperatures. E. Mach.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.